Beurling regular variation, Bloom dichotomy, and the Gołąb-Schinzel functional equation

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To Anatole Beck on his 83$^{\mathrm{rd}}$ *birthday.*

Abstract. The class of 'self-neglecting' functions at the heart of Beurling slow variation is expanded by permitting a positive asymptotic limit function $\lambda(t)$, in place of the usual limit 1, necessarily satisfying the following 'self-neglect' condition:

$$\lambda(x)\lambda(y) = \lambda(x + y\lambda(x)),$$

known as the *Gołąb-Schinzel functional equation*, a relative of the Cauchy equation (itself also central to Karamata regular variation). This equation, due independently to Aczél and Gołąb, occurring in the study of one-parameter subgroups, is here accessory to the λ-Uniform Convergence Theorem (λ-UCT) for the recent, flow-motivated, 'Beurling regular variation'. Positive solutions, when continuous, are known to be $\lambda(t) = 1 + at$ (below a new, 'flow', proof is given); $a = 0$ recovers the usual limit 1 for self-neglecting functions. The λ-UCT allows the inclusion of Karamata multiplicative regular variation in the Beurling theory of regular variation, with $\lambda(t) = 1 + t$ the relevant case here, and generalizes Bloom's theorem concerning self-neglecting functions.

Keywords: Beurling regular variation, self-neglecting functions, uniform convergence theorem, category-measure duality, Bloom dichotomy, Gołąb-Schinzel functional equation.

Classification: 26A03; 33B99, 39B22, 34D05; 39A20

1. Regular variation, self-neglecting and Beurling functions

The Karamata theory of regular variation studies functions $f : \mathbb{R} \to \mathbb{R}$ with

$$f(tx)/f(x) \to g(t) \text{ as } x \to \infty \qquad \forall t, \tag{RV}$$

(and f is *slowly varying* if $g = 1$), or equivalently in isomorphic additive form

$$h(x + t) - h(x) \to k(t) \qquad \forall t, \tag{RV_+}$$

for $h : \mathbb{R}_+ \to \mathbb{R}_+$. Our reference for regular variation is [BinGT] (BGT below). The Beurling theory of slow variation, originating in Beurling's generalization (for which see [Pet] and [Moh] – cf. [BinO6]) of the Wiener Tauberian Theorem, studies functions f with

$$f(x + t\varphi(x))/f(x) \to 1 \qquad \forall t, \tag{BSV}$$

where φ is *positive* (on $\mathbb{R}_+$) and itself satisfies (BSV) with φ for f, i.e.

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \qquad \forall t; \tag{BSV_φ}$$

call such a φ 'Beurling-slow'. If convergence here is locally uniform in t, then φ is said to be *self-neglecting* (BGT §2.11; cf. [Moh], [Pet]), i.e.

$$\varphi(x + u\varphi(x))/\varphi(x) \to 1, \text{ locally uniformly in } u. \tag{SN}$$

Bloom [Blo] shows that a *continuous* Beurling-slow φ satisfies SN, and that $\varphi(x) = o(x)$. More generally, a Baire/measurable φ with a little more regularity (e.g. the Darboux property) satisfies SN; this may be viewed as a *Bloom dichotomy*: a Beurling-slow function is either self-neglecting or pathological (see [BinO7], or Section 5).

Although (BSV) includes via $\varphi = 1$ the Karamata *additive* slow version (i.e. RV_+ with $k = 0$), it excludes $\varphi(x) = x$ and the *multiplicative* Karamata format (RV), which, but for $\varphi(x) = o(x)$, it would capture. So one aim is to expand the notion of self-neglect to allow direct specialization to the multiplicative Karamata form; our approach is motivated by recent work extending Beurling slow variation to Beurling regular variation. We recall from [BinO8] that f is *φ-regularly varying* if as $x \to \infty$

$$f(x + t\varphi(x))/f(x) \to g(t), \qquad \forall t, \tag{BRV}$$

and φ, the *auxiliary function,* is *self-neglecting.* In Theorem 2 below we show that the multiplicative Karamata theory can be incorporated in a Beurling framework, but only if one replaces the limit 1 occurring above in (BSV_φ) with a more general limit $\lambda(t)$ – yielding what we call *self-equivarying* functions with limit λ (definition below); exactly as with its Beurling analogue, Karamata multiplicative theory then takes its uniformity from the uniformity possessed by $\varphi(x) = x$. The case $\varphi(x) = 1$ specializes to the Uniform Convergence Theorem of Karamata additive theory (UCT) – see BGT §1.2.

The recent Beurling theory of regular variation was established using the affine combinatorics of [BinO7], where SN was deduced for Baire/measurable φ under various side-conditions including the Darboux property, more general than Bloom's continuity (as above) and more natural, since it implies continuous orbits for the underlying differential flows of Beurling variation in the measure case – defined by $\dot{x}(t) = \varphi(x(t))$. (See also its natural occurrence in [Jab].) In [BinO8] it is shown that the uniformity in (SN) passes 'out' to uniformity in (BRV) and noted that conversely if $\varphi(x) = o(x)$, then the assumption of uniformity, but only in (BRV), passes 'in' the uniformity to the auxiliary function, when both are measurable or both have the Baire property (briefly: are Baire) – see BGT §3.10 for the 'φ monotonic' paradigm. Our methods focus on the in-out transfer of uniformity by considering a natural context of asymptotic equivalence, one that includes the Karamata multiplicative theory directly.

Definitions. Say that f and g are *Beurling φ-equivarying*, or f is *Beurling φ-equivarying with g*, if

$$f(x + t\varphi(x))/g(x) \to 1 \text{ as } x \to \infty, \text{ for all } t > 0. \qquad (BE_\varphi)$$

Call f, g *uniformly Beurling φ-equivarying* if

$$f(x + t\varphi(x))/g(x) \to 1 \text{ as } x \to \infty, \text{ on compact sets of } t > 0. \qquad (UBE_\varphi)$$

For appropriate φ (as below), these actually yield equivalence relations on functions satisfying (BSV); indeed transitivity follows from

$$\frac{f(x + t\varphi(x))}{h(x)} = \frac{f(x + t\varphi(x))}{g(x)} \cdot \frac{g(x)}{g(x + t\varphi(x))} \cdot \frac{g(x + t\varphi(x))}{h(x)}. \qquad (1)$$

Proceeding as in (1) justifies the preferred *symmetric* terminology in an apparently asymmetric context; we omit the routine details, save to assert:

Proposition 1 (Symmetry). *For f, g satisfying (BSV):*
(i) *if f is Beurling φ-equivarying with g, then g is Beurling φ-equivarying with f;*
(ii) *similarly for f is uniformly Beurling φ-equivarying with g.*

The two equivalence relations call for a study of 'self-equivalence' in Beurling regular variation terms – henceforth termed *equivariation*, or *equivariance*.

Definitions. (i) For $\varphi(x) = O(x)$, say that $\varphi > 0$ is *self-equivarying,* $\varphi \in SE$, with *limit* λ, if uniformly

$$\varphi(x + t\varphi(x))/\varphi(x) \to \lambda(t) \text{ as } x \to \infty, \text{ on compact sets of } t > 0. \qquad (SE_\lambda)$$

(ii) For $\varphi(x) = O(x)$, say that $\varphi > 0$ is *weakly self-equivarying,* $\varphi \in WSE$ *with limit* λ, if pointwise

$$\varphi(x + t\varphi(x))/\varphi(x) \to \lambda(t) \text{ as } x \to \infty, \text{ for all } t > 0. \qquad (WSE)$$

(iii) For (positive) $\varphi \in WSE$, we set for $t > 0$

$$\lambda_\varphi(t) := \lim_{x\to\infty} \frac{\varphi(x + t\varphi(x))}{\varphi(x)}, \text{ and } \lambda_\varphi(0) := 1. \qquad (2)$$

Preservation of SE and SN under equivariance (see Th. 5), and characterizing the limit λ_φ above for self-equivarying φ (see Th. 0) thus call for attention. The latter is linked to the *Cauchy functional equation* for additive functions (for which see [Kucz], [AczD]), which already plays a key role in determining the index theory of Karamata regular variation – see [BinO1]. Here, for Beurling regular variation, there is an analogous functional equation satisfied by the limit functions λ_φ, namely the *Gołąb-Schinzel equation*

$$\lambda(x)\lambda(y) = \lambda(x + y\lambda(x)) \ (\forall x, y), \qquad (GS)$$

first considered by Aczél [Acz] in work on geometric objects and independently by Gołąb in the study of 3-parameter affine subgroups of the plane. We refer to it as the functional *equation of self-neglect* and solutions positive on $\mathbb{R}_+$ as *Beurling functions.* Its additive equivalent for $\kappa = \kappa_\lambda := \log \lambda$ is

$$\kappa(x + y\lambda(x)) - \kappa(x) = \kappa(y), \text{ or } \Delta_y^\lambda \kappa(x) = \kappa(y) \qquad (3)$$

(in mixed form), where

$$\Delta_y^\lambda \kappa(x) := \kappa(x + y\lambda(x)) - \kappa(x), \qquad (4)$$

stresses the underlying 'Beurling difference-operator'. Viewing inputs as time, λ represents a *local time-change* – for connections here to the theory of flows see [Bec2, Ch. 4]; cf. [BinO8], the earlier [Ost], and [BinO1].

Aczél originally observed in 1957 that the non-zero differentiable solutions of an equivalent form of (GS) take the form $1 + ax$; independently, a general analysis of its solutions was undertaken by Gołąb in collaboration with Schinzel in 1959 ([GołS]) and was amplified in 1965 by Popa's semi-group perspective via $x *_\lambda y := x + y\lambda(x)$ [Pop], surprisingly consonant with Beurling's 'generalized' convolution approach to the Wiener Tauberian theorem; but a classification of measurable solutions had to wait till Wołodźko [Woł] in 1968, with some of the complex-variable context being extended in 1989 by Baron [Bar]. This is reviewed in [AczG]; for a recent text-book account see [AczD, Ch. 19] or the more recent survey [Brz2], which includes generalizations of (GS) and a discussion of applications in algebra, meteorology and fluid mechanics – see for instance [KahM]. The key concept in this literature is micro-periodicity of solution functions (i.e. whether functions have arbitrarily small periods, and so a dense set of periods), an idea due to Burstin in 1915 [Bur] and Łomnicki in 1918 [Lom] (a measurable micro-periodic function is constant modulo a null set – see e.g. [BrzM, Th. Prop. 2]). Of interest is Theorem A below due to Popa, based on the Steinhaus subgroup theorem applied to the set of periods (an additive subgroup). Though the proof is given in the measure case, the category case is similar. Recall that 'quasi everywhere' means 'off a negligible set', be it meagre or null.

Theorem A ([Pop, Th. 2] measure case, [Brz1] Baire case; cf. [Mur2] Christensen-measurable case). *Every measurable/Baire solution of the Gołąb-Schinzel equation is either continuous or quasi everywhere zero.*

Our interest is only in solutions that are positive on $\mathbb{R}_+$, so when they are Baire or measurable (as will be the case for λ_φ for Baire/measurable φ), the Beurling functions are necessarily continuous and of the form $1+ax$ with $a \geq 0$. In view of their importance we give a new analysis (in §6) of this affine representation via the topological dynamics approach that underpins regular variation.

A brief comparison with Cauchy's exponential equation:

$$f(x)f(y) = f(x+y) \ (\forall x, y), \qquad (CFE)$$

is helpful here; just as its continuous solutions are indeed the exponentials

e^{at}, those of (GS) are the linear part[1] of the same exponential: $1+at$. Recall also that additive functions if continuous are linear and so differentiable; they are continuous if Baire (Banach [Ban, Ch. I, §3, Th. 4]), if measurable (Fréchet), if bounded on a non-null measurable set (Ostrowski's Theorem, refining Darboux's result for intervals), or on a non-meagre Baire set (Mehdi [Meh]); see [Kucz], or the more recent account in [BinO5]. Such automatic continuity results are mirrored in Beck's 'algebraic flows' in a metric space, which when bounded by a monotone function of the flow's distance from some set K are continuous at points of K ([Bec2, Th. 1.65]). The latter approach motivates a new proof of the Aczél-Gołąb-Schinzel representation (in §6) and perhaps explains why (GS) has analogues, though not exact replicates, and possesses similarly to (CFE) unbounded discontinuous solutions, granted the existence of a Hamel basis (see [GolS]). [AczG] notes that $\mathbf{1}_{\mathbb{Q}}$, the indicator of the rationals (Dirichlet's function), is a measurable, bounded discontinuous solution to (GS) – a contrast to Ostrowski's Theorem.

Remarks. 1. If φ is Baire, then λ_φ is Baire, being the limit of functions $\varphi_n(t) := \varphi(n+t\varphi(n))/\varphi(n)$, for $n \in \mathbb{N}$, which are Baire as each $t \to n+t\varphi(n)$ is a homeomorphism. Similarly for measurability.
2. If λ_φ is continuous in (SE), then for $\varepsilon > 0$ and $t_n \to t$

$$|\varphi(x + t_n\varphi(x))/\varphi(x) - \lambda_\varphi(t)| < \varepsilon, \text{ for large enough } n \text{ and } x. \qquad (SSE)$$

This *strong self-equivariance* condition (SSE) could be adopted in place of (SE), with continuity of λ immediate – motive enough to study (SE).
3. For $\lambda(t) \equiv 1$, (SE_λ) differs from (SN) in requiring $O(x)$ rather than $o(x)$.
4. If $\varphi(x) = ax$ with $a > 0$, then $\varphi(x) = O(x)$ and we have an *affine* form:[2]

$$\varphi(x + t\varphi(x))/\varphi(x) = a(x + atx)/ax = 1 + at = \lambda_\varphi(t).$$

We now establish the significance of (GS) for Beurling regular variation.

[1] Interestingly, (GS) implies a self-differential property:

$$\frac{d}{du}\lambda(u\lambda(t) + t) = \lambda(u\lambda(t) + t) \cdot \frac{\lambda'(u)}{\lambda(u)}.$$

[2] Affine functions $f : \mathbb{R}^d \to \mathbb{R}$ are termed *linear* in [Kucz, §7.7]. This usage sits well with the context of $\mathbb{R}$ as a field over $\mathbb{Q}$, to which the Beurling equation seems less suited.

Theorem 0 (A Characterization Theorem). *For Baire/measurable $\varphi \in SE$ the limit function λ_φ satisfies (GS), so is continuous, and if positive has the form $\lambda(t) = 1 + at$.*

Furthermore, $a \geq 0$ is required for λ_φ when φ satisfies the order condition $\varphi(x) = O(x)$. Also, up to re-scaling, there are only the two limits λ_φ: small-order limit $\lambda(t) \equiv 1$ and large-order limit $\lambda(t) \equiv 1 + t$.

Proof. Suppose that $\varphi \in SE$; writing $y := x + u\varphi(x)$ and $s = v\lambda_\varphi(u)$ note that

$$\frac{\varphi(x + (u+s)\varphi(x))}{\varphi(x)} = \frac{\varphi(y + s\frac{\varphi(x)}{\varphi(x+u\varphi(x))}\varphi(y))}{\varphi(y)} \cdot \frac{\varphi(x + u\varphi(x))}{\varphi(x)}. \tag{5}$$

The left-most and right-most terms tends to $\lambda_\varphi(u+s)$ and $\lambda_\varphi(u)$ respectively. Now $s\varphi(x)/\varphi(x + u\varphi(x)) \to s/\lambda_\varphi(u) = v$. Let $x \to \infty$ to get

$$\lambda_\varphi(u+s)/\lambda_\varphi(u) = \lambda_\varphi(v),$$

as required. For φ Baire/measurable, λ_φ is Baire/measurable and satisfies (GS) so, by Theorem A, is continuous. By the results referred to above of Gołąb and Schinzel, and Wołodźko (or see [AczD, Ch. 19 Prop.1]), we conclude that

$$\lambda_\varphi(t) = 1 + at.$$

The condition $\varphi(t) = O(t)$ yields $a \geq 0$.

Given $\varphi \in WSE$, re-scaling to $\psi(t) = \varphi(t)/b$ with $b > 0$ yields

$$\lambda_\varphi(t) = \lim_x \varphi(x + bt\varphi(x)/b)/\varphi(x) = \lim_x \psi(x + bt\psi(x))/\psi(x) = \lambda_\psi(bt),$$

i.e. $\lambda_{\varphi/b}(bt) = \lambda_\varphi(t)$. So if $\lambda_\varphi(t) = 1 + at$, taking $b = 1/a$ yields $\lambda_{a\varphi}(t/a) = 1 + t$. □

Remark. We note for completeness of §6 that, for λ differentiable, differentiating (GS) w.r.t. y yields $\lambda'(y) = \lambda'(x + y\lambda(x))$ and in particular $\lambda'(x) = \lambda'(0)$, whence $\lambda(x) = 1 + ax$, as $\lambda(0) = 1$.

Corollary (Representation for SE). *For Baire/measurable $\varphi \in SE$ with positive limit λ_φ the function $\psi(x) := \varphi(x)/\lambda_\varphi(x)$ is self-neglecting and so*

$$\varphi(x) \sim \lambda_\varphi(x) \int_0^x e(u)du \text{ for some continuous } e \text{ with } e \to 0.$$

Proof. By Theorem 0, we may assume that $\lambda_\varphi(x) = 1 + ax$ for some $a > 0$, otherwise there is nothing to prove. So $\psi(x) = O(1)$, as $\varphi(x) = O(x)$. Fix $t > 0$; then $s_x := t\psi(x)/\varphi(x) \to 0$. Now $\lambda_\varphi(x)/\lambda_\varphi(x + t\psi(x)) \to 1$, so

$$\psi(x+t\psi(x))/\psi(x) = \varphi(x+s_x\varphi(x))/\varphi(x)\cdot\lambda_\varphi(x)/\lambda_\varphi(x+t\psi(x)) \to \lambda_\varphi(0) = 1.$$

So $\psi \in SN$ and the representation follows from a result of Bloom (see [Blo]; cf. [BinO8]).

2. Combinatorial preliminaries

We summarize from [BinO8] the combinatorial framework needed here: Baire and measurable cases are handled together by working bi-topologically, using the Euclidean topology in the Baire case (the primary case) and the density topology in the measure case; see [BinO2], [BinO5], [BinO4]. We work in the affine group $\mathcal{A}ff$ acting on $(\mathbb{R}, +)$ using the notation

$$\gamma_n(t) = c_n t + z_n,$$

where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$, as in Theorem B below. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either the Euclidean topology, or the density topology. We recall the following result from [BinO7].

Theorem B (Affine Two-sets Theorem). *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible (measurable/Baire), then for quasi all $b \in B$ there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{\gamma_m(b) = c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

As in [BinO8], Theorem 1 below needs only the case $c = 1$; however, Theorem 3 needs the case $c \neq 1$.

3. Uniform Convergence Theorem

This section closely mirrors [BinO8, §4] in verifying the generalization needed here; some care is needed to distinguish SE from SN, likewise UBE_φ involving 1 as limit – from WSE involving a general limit λ. Our convention is to write $f_{\mathrm{N}} := f$ and $f_{\mathrm{D}} = g$, ("N for numerator, D for denomintor") and also

$$h := \log\varphi \qquad h_{\mathrm{N}} := \log f_{\mathrm{N}} \text{ and } h_{\mathrm{D}} := \log f_{\mathrm{D}}.$$

Definition. (i) For $\varphi \in SE$ call $\{u_n\}$ with limit u a *1-witness sequence at* u (for non-uniformity in f_{N} *over* f_{D}) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n with

$$|h_{\mathrm{N}}(x_n + u_n\varphi(x_n)) - h_{\mathrm{D}}(x_n)| > \varepsilon_0 \qquad \forall\, n \in \mathbb{N}. \tag{6}$$

(ii) For $\varphi \in WSE$ call $\{u_n\}$ with limit u a *WSE-witness sequence at u* (for non-uniformity in φ) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n with

$$|h(x_n + u_n\varphi(x_n)) - h(x_n) - \kappa(u)| > \varepsilon_0 \qquad \forall\, n \in \mathbb{N}. \tag{7}$$

Call $\{u_n\}$ with limit u a *divergent WSE-witness sequence* if also

$$h(x_n + u_n\varphi(x_n)) - h(x_n) \to \pm\infty.$$

So divergence gives a special type of WSE-witness sequence.
Below, *uniform near* a point u means 'uniformly on sequences converging to u' – equivalent to local uniformity at u (on compact neighbourhoods of u).

Lemma 1 (Shift Lemma: uniformity preservation under shift).
(i) *Let $\varphi \in SE$. For any u, convergence in (BE_φ) is uniform near $t = 0$ iff it is uniform near $t = u$.*
(ii) *Let $\varphi \in WSE$ with limit λ_φ: for any u, convergence in (WSE) is uniform near $t = 0$ iff it is uniform near $t = u$.*

Proof. Since in case (i) $h_{\mathrm{N}}(x_n + u\varphi(x_n)) - h_{\mathrm{D}}(x_n) \to 0$ and in case (ii) $h(x_n + u\varphi(x_n)) - h(x_n) - \kappa(u) \to 0$ argue routinely, as in [BinO7]. □

Theorem 1 follows from the argument presented in [BinO8, Th. 2] with minimal amendments, so a sketch suffices; the detailed proof of Theorem 3 below (responding to the presence of λ in WSE) is a paradigm for the SE case here.

Theorem 1 (λ-Uniform convergence theorem, λ-UCT). *For $\varphi \in SE$ with limit $\lambda = \lambda_\varphi$, if f, g, φ have the Baire property (are measurable) and satisfy (BE_φ), then they satisfy (UBE_φ).*

Proof. Suppose otherwise. By Theorem A the limit λ_φ is continuous. Now begin as in [BinO8, Th. 2]; let u_n be a 1-witness sequence for the non-uniformity of f over g. For some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (6). By the

Shift Lemma (i), we may assume that $u = 0$. So we will write z_n for u_n. As φ is self-equivarying for any $\varepsilon > 0$ and with $K := \{z_n : n = 0, 1, 2, ..\}$ (compact) for large enough n

$$|h(x_n + z_n\varphi(x_n)) - h(x_n) - \kappa(z_n)| \leq \varepsilon \qquad \forall\, n \in \mathbb{N}.$$

But κ is continuous, so that $\kappa(z_n) \to \log\lambda(0) = 0$, and so

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow 1 = \lambda_\varphi(0). \tag{8}$$

Write $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$, and

$$|h_{\mathrm{N}}(y_n) - h_{\mathrm{D}}(x_n)| \geq \varepsilon_0.$$

Continue verbatim as in [BinO8], applying Theorem B to $\gamma_n(s) := c_n s + z_n$ to derive a contradiction to (6). □

As an immediate corollary we have:

Theorem 2 (Beurling and Karamata UCT). *For $\varphi \in SN$, if f, φ have the Baire property (are measurable) and satisfy (BRV), then they satisfy (BRV) locally uniformly.*

For $\varphi(x) = x$, if f has the Baire property (is measurable) and satisfies (RV), then f satisfies

$$f(tx)/f(x) \to g(t), \text{ as } x \to \infty \text{ locally uniformly in } t \tag{RV}$$

Proof. In Theorem 1, take $g = f$. □

Theorem 1 invites an extension of Beurling regular variation based on $\varphi \in SE$, i.e. beyond SN. That extension yields only multiplicative Karamata regular variation – because, by Theorem 0, up to rescaling ("in t"), there is only one 'canonical' alternative beyond SN, namely $\lambda_\varphi(t) = 1 + t$, occuring e.g. for $\varphi(x) = x$. Here one has $f(x + t\varphi(x))/f(x) = f(x(1 + t))/f(x)$ so the unit shift on t below is inevitable.

Theorem 1′ (Extended regular variation). *For $\varphi \in SE$ if f, φ have the Baire property (are measurable), $\lambda_\varphi(t) = 1 + t$, and f satisfies for $t > 0$*

$$f(x + t\varphi(x))/f(x) \to \gamma(t),$$

then $\gamma(t) = (1+t)^\rho$ for some $\rho \in \mathbb{R}$.

Proof. In Theorem 1, again with $g = f$, (UBE_φ) holds. So for $\gamma(t) := \lim f(x + t\varphi(x))/f(x)$, writing $y = x + s\varphi(x)$ and noting that $t\varphi(x)/\varphi(y) \to v := t/\lambda_\varphi(s)$, by (UBE_φ) one has

$$\begin{aligned}\gamma(s+t) &= \lim \frac{f(x+(s+t)\varphi(x))}{f(x)} = \lim \frac{f(y + [t\varphi(x)/\varphi(y)]\varphi(y))}{f(y)} \cdot \frac{f(x+s\varphi(x))}{f(x)} \\ &= \gamma(v)\gamma(t)\end{aligned}$$

(as $y \to \infty$ when $x \to \infty$), or

$$\gamma(u + v\lambda_\varphi(u)) = \gamma(v)\gamma(t),$$

where $\lambda_\varphi(t) = 1 + t$. Putting $G(t) = \gamma(t-1)$, $x = 1+u, y := 1+v$, one has

$$G(xy) = \gamma(u + v + uv) = \gamma(u)\gamma(v) = G(x)G(y).$$

As G is Baire/measurable, $G(x) = x^\rho$ for some ρ (see [AczD, Ch. 3]), so $\gamma(t) = G(1+t) = (1+t)^\rho$. □

4. Stability properties of Beurling functions

There is a literature surrounding (GS) and its generalizations devoted to stability properties in the sense of Heyers-Ulam-Rassias – for the general context and the literature concerned with (GS), initiated by Ger and his collaborators, see for example [CharBKR], cf. [ChuT]. We pursue a related agenda but motivated by the regular variation view of the interplay between $\varphi \in WSE$ and λ_φ. We begin with a rigidity property noting first a formula, an instance of which is the *doubling formula* $\lambda(2t) = \lambda(t/\lambda(t))\lambda(t)$. We omit the routine proof.

Lemma 2 (Internal time-change). *For λ satisfying (GS) the internal time-change $\mu(t) := \lambda(\beta t)$ yields a solution to (GS). Also one has*

$$\mu(t) = \lambda(\beta t) = \lambda(t)\lambda(\alpha t/\lambda(t)), \text{ with } \alpha := \beta - 1.$$

Proposition 2 (Slow time-changing). *For $\lambda \in SE$ and $w(.)$ Baire satisfying*

$$\lim_{x\to\infty} \frac{w(x + u\lambda(x))}{w(x)} = 1 \text{ and } \lim_{x\to\infty} w(x) = 1 + \alpha, \qquad \forall u,$$

the time-changed function $\mu(x) := \lambda(x)w(x)$ *is a solution of* (GS) *iff*

$$w(t) = \lambda\left(\alpha t/\lambda(t)\right).$$

In particular, for $\beta = 1$*, we have* $w(t) = 1$.

Proof. Put $\mu(x) = \lambda(x)w(x)$; if μ is a solution of (GS), then $\mu(t) = \mu(x + t\mu(x))/\mu(x)$. Substituting into this identity,

$$\frac{\lambda(x + t\lambda(x)w(x))}{\lambda(x)} \frac{w(x + t\lambda(x)w(x))}{w(x)} = \lambda(t)w(t).$$

Using $\lambda(t) = \lambda(x + t\lambda(x))/\lambda(x)$ twice, we have

$$w(t) = \frac{\lambda(x + t\lambda(x) + t[w(x) - 1] \cdot \frac{\lambda(x)}{\lambda(x+t\lambda(x))} \cdot [\lambda(x + t\lambda(x))])}{\lambda(x + t\lambda(x))} \cdot \frac{w(x + t\lambda(x)w(x))}{w(x)}.$$

Put $y := x + t\lambda(x)$ and $u(x) = t(w(x) - 1)/\lambda(t)$; then

$$\frac{\lambda(y + u(x)\lambda(y))}{\lambda(y)} \frac{w(x + [tw(x)]\lambda(x))}{w(x)} = w(t),$$

or

$$\frac{\lambda(y + u(x)\lambda(y))}{\lambda(y)} \frac{w(x + [tw(x)]\lambda(x))}{w(x)} = w(t), \text{ i.e. } \lambda(u(x))\frac{w(x + [tw(x)]\lambda(x))}{w(x)} = w(t).$$

As $\lambda \in SE$, if w is Baire and λ-slowly varying and bounded, then by UCT

$$\frac{w(x + [tw(x)]\lambda(x))}{w(x)} \to 1.$$

So if $w(x) \to 1 + \alpha$, then $u(x) \to t\alpha/\lambda(t)$, and so $w(t) = \lambda\left(\alpha t/\lambda(t)\right)$.

For the converse, apply Lemma 2. □

Example. Taking $\lambda(t) = 1 + t$, we have $w(t) = 1 + \alpha t/(1 + t)$ and

$$\mu(t) = (1 + t)\left(1 + \frac{\alpha t}{1 + t}\right) = (1 + t) + \alpha t = 1 + (1 + \alpha)t.$$

Theorem 3 enables an extension of Bloom's Theorem (see §1 and 5) with WSE replacing the original 'slow Beurling'. Analogous to the Divergence

Theorem of [BinO7], but more subtle, an extra twist calls for a detailed proof. It should be borne in mind that here λ_φ is not known to satisfy (GS); that will be deduced later in Th. 4. The continuity assumption at 0 seems an inevitable 'connection' of the two parts of the definition (2).

Theorem 3 (Divergence Theorem – Baire/measurable). *For φ Baire/measurable in WSE with limit λ_φ continuous at 0: if u_n with limit u is a WSE-witness sequence to non-uniformity of φ over φ, then either u_n is a divergent witness sequence, or for some divergent sequence x_n*

$$\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \to \lambda_\varphi(u).$$

Proof. Begin as in the proof of Theorem 2, except that here $h_{\mathrm{N}} = h_{\mathrm{D}} = h = \log \varphi$. Let u_n with limit u be a WSE-witness sequence to non-uniformity of φ over φ, with limit λ; for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (7) with $\kappa = \log \lambda$. By the Shift Lemma (ii), we may assume that $u = 0$. So we will write z_n for u_n. That is, with $y_n := x_n + z_n\varphi(x_n)$,

$$|h(y_n) - h(x_n)| > \varepsilon_0.$$

Note that $y_n = x_n(1 + z_n\varphi(x_n)/x_n)$ is divergent. Assume the non-divergence of $\{h(y_n) - h(x_n)\}$. Consider any convergent subsequence; we show its limit is 0, by contradiction. Working down a subsequence, suppose that

$$c_n := \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \longrightarrow c \in (0, \infty), \text{ with } c \neq 1. \tag{9}$$

As $|h(y_n) - h(x_n)| > \varepsilon_0$, passing to the limit we obtain

$$\log c \geq \varepsilon_0 > 0.$$

Choose η_0 with $0 < \eta_0 < \frac{1}{2}\log c$ and let $\eta = \eta_0/6$.

Suppose now that κ has the Baire property and is continuous on a co-meagre set S – see [Oxt, Th. 8.1] or [Kur, §28]. Take $T_0 := S$, set inductively $T_{n+1} := cT_n \cap T_n$ and $T_{-(n+1)} := c^{-1}T_{-n} \cap T_{-n}$, and put $T := \bigcap_{n=-\infty}^{+\infty} T_n$. Then $ct \in T$ and $c^{-1}t \in T$ for $t \in T$: each T_n and so T is co-meagre. So the restriction $\kappa|T$ is continuous on T.

By assumption there is $\delta_0 > 0$ such that for $s \in (0, \delta_0)$

$$|\kappa(c^{-1}s) - \kappa(s)| < \eta.$$

For $x = \{x_n\}$, working in T, put

$$V_n^x(\eta) := \{s \in T : |h(x_n+s\varphi(x_n))-h(x_n)-\kappa\,(s)\,| \leq \eta\},\ \ H_k^x(\eta) := \bigcap\nolimits_{n \geq k} V_n^x(\eta),$$

and likewise for $y = \{y_n\}$. These are Baire sets, and

$$T = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{10}$$

as $\varphi \in WSE$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers $T \cap (0, \delta_0)$. So for some k the set $H_k^x(\eta) \cap (0, \delta_0)$ is non-negligible. As $c^{-1}H_k^x(\eta)$ is non-negligible, so is $c^{-1}H_k^x(\eta) \cap T$ as well as $H_k^x(\eta) \cap cT$ and $H_k^x(\eta) \cap T$; by (10), for some l the set

$$B := c^{-1}[H_k^x(\eta) \cap (0, \delta_0)] \cap H_l^y(\eta)$$

is also non-negligible. Take $A := T \cap H_k^x(\eta)$; then $B \subseteq H_l^y(\eta)$ and $cB \subseteq A$ with A, B non-negligible. Applying Theorem B of §2 to the maps $\gamma_m(s) := c_n s + z_n$ with $c = \lim_n c_n$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta),$$

and as $bc \in (0, \delta_0)$

$$|\kappa(b) - \kappa(bc)| < \eta.$$

That is, as $B \subseteq H_l^y(\eta)$,there is $b \in H_l^y(\eta)$ and an infinite $\mathbb{M}_t$ such that, on adding u

$$\{\gamma_m^u(b) := c_m b + z_m : m \in \mathbb{M}_t\} \subseteq H_k^x(\eta).$$

In particular, for this b and $m \in \mathbb{M}_b$ with $m > k, l$ one has

$$b \in V_m^y(\eta) \text{ and } \gamma_m(b) \in V_m^x(\eta).$$

As $t := cb \in T$ and $\gamma_m^u(b) \in T$, we have by continuity of $\kappa|T$ at t, since $\gamma_m(b) \to cb$, that for all m large enough

$$|\kappa(t) - \kappa(\gamma_m(b))| \leq \eta. \tag{11}$$

Fix such an m. As $\gamma_m(b) \in V_m^x(\eta)$,

$$|h(x_m + \gamma_m(b)\varphi(x_m)) - h(x_m) - \kappa(\gamma_m(b))| \leq \eta. \tag{12}$$

But $\gamma_m(b) = c_m b + z_m = z_m + b\varphi(y_m)/\varphi(x_m)$, so

$$x_m + \gamma_m(b)\varphi(x_m) = x_m + z_m\varphi(x_m) + b\varphi(y_m) = y_m + b\varphi(y_m),$$

'absorbing' the affine shift component of $\gamma_m(b)$ into y. So, by (12),

$$|h(y_m + b\varphi(y_m)) - h(x_m) - \kappa(\gamma_m(b))| \leq \eta.$$

But $b \in V_m^y(\eta)$, so

$$|h(y_m + b\varphi(y_m)) - h(y_m) - \kappa(b))| \leq \eta.$$

Using the triangle inequality, and combining the last two inequalities with (11), we have

$$\begin{aligned} |h(y_m) - h(x_m)| &\leq |h(y_m + b\varphi(y_m)) - h(y_m) - \kappa(b)| \\ &\quad +|\kappa(b) - \kappa(cb)| + |\kappa(cb) - \kappa(\gamma_m^u(b))| \\ &\quad +|h(y_m + t\varphi(y_m)) - h(x_m) - \kappa(\gamma_m^u(b))| \\ &\leq 4\eta < \eta_0. \end{aligned}$$

For large m one has $\log c - \eta_0 < h(y_m) - h(x_m) < \log c + \eta_0$, so for any one such large m we have $\log c - \eta_0 < h(y_m) - h(x_m) < \eta_0$, that is, $\log c < 2\eta_0$ contradicting the choice of η_0. Thus $c = 1$.

Now suppose that κ is measurable. Proceed as before, but now apply Luzin's Theorem ([Oxt], Ch. 8) to select $T \subseteq [c, 2c] \cup [1, 2]$ such that $|T \cap [1, 2]| > 2/3$ and $|T \cap [c, 2c]| > 3c/4$ with $\kappa|T$ continuous on T. As before, put

$$V_n^x(\eta) := \{s \in T : |h(x_n+s\varphi(x_n))-h(x_n)-\kappa(s)| \leq \eta\},\ H_k^x(\eta) := \bigcap\nolimits_{n\geq k} V_n^x(\eta),$$

and likewise for $y = \{y_n\}$. These are measurable sets, and

$$T = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{13}$$

since $\varphi \in WSE$. The increasing sequence of sets $\{H_l^y(\eta)\}$ covers $T \cap [c, 2c]$. So $|(T \cap [c, 2c]) \cap H_k^x(\eta)| > 2|T \cap [c, 2c]|/3$ for some k. So in particular $H_k^x(\eta)$ is non-null, and furthermore, $|T \cap [c, 2c] \backslash H_k^x(\eta)| < |T \cap [c, 2c]|/3 < c/3$. So $|[1, 2] \backslash c^{-1}H_k^x(\eta)| < 1/3$; but $|T \cap [1, 2]| > 2/3$, so $|c^{-1}H_k^x(\eta) \cap [1, 2]| > 0$; by (13), for some l the set

$$B := c^{-1}H_k^x(\eta)) \cap H_l^y(\eta)$$

is also non-null. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and $cB \subseteq A$ with A, B non-null. From here continue as in the Baire argument. $\square$

5. The extended Bloom dichotomy.

The preceding section implies the *Bloom dichotomy* – that φ Beurling-slow (i.e. φ with $\lambda_\varphi = 1$) is either self-neglecting or pathological – extends to WSE: when $\varphi \in WSE$ either $\varphi \in SE$, or φ is 'pathological'. (For other occurrences of dichotomy in this area see [BinO3,4,5].) Indeed, $\varphi \in WSE$ says merely that the limit function λ_φ is well-defined, but nothing about whether λ_φ satisfies (GS). However, if λ_φ is *continuous at the origin* and φ has just the kind of regularity considered in the Generalized Bloom Theorem of [BinO7], then in fact $\varphi \in SE$, so that λ_φ satisfies (GS) and takes a simple form. This brings to mind, as an analogy, Lévy's Continuity (or Convergence) Theorem, see [Wil, Ch.18], or [Dud, 9.8.2], that if a sequence of characteristic functions converges pointwise to a limit function which is *continuous at the origin*, then that limit is itself a characteristic function; the continuity assumption is critical, as Bochner's theorem asserts the converse: λ a positive-definite function, normalized so that $\lambda(0) = 1$, and continuous at the origin is a characteristic function (cf. [Rud,1.4.3]).

Theorem 4 (Bloom's Theorem for weak self-equivariance). *For $\varphi \in WSE$ with limit function λ_φ continuous at 0 and $\varphi(x) = O(x)$, if φ is Baire/measurable and has any of the following properties:*
(i) *φ has the Darboux property (in particular, φ is continuous),*
(ii) *$\varphi(x)$ has bounded range on $(0, \infty)$,*
(iii) *$\varphi(x)/x$ is bounded in $(0, \infty)$,*
(iv) *$\varphi(x)$ is increasing in $(0, \infty)$,*
then $\varphi \in SE$ and so λ_φ is continuous.

Proof. Apply Theorem 3 and use the Darboux property as in the Beurling-Darboux UCT of [BinO7, Th. 4] to argue as with Bloom's Theorem that there are no divergent witness sequences. $\square$

Theorem 5. (i) *For $\varphi \in SE$, if $\psi > 0$ is smooth, Beurling-slow and Beurling φ-equivarying with φ, then $\psi \in SE$ and φ is ψ-equivarying with ψ, and likewise for SN, mutatis mutandis, so in particular:*
(ii) *For $\varphi \in SN$, if $\psi > 0$ is smooth and Beurling φ-equivarying with φ, then $\psi \in SN$.*

Proof. Notice first that for any fixed $u > 0$, we have

$$\psi(x)/\varphi(x) = \psi(x)/\psi(x+u\varphi(x)) \cdot \psi(x+u\varphi(x))/\varphi(x) \to 1,$$

since ψ satisfies (BSV) and ψ is Beurling φ-equivarying with φ. So one has $\psi(x) = O(x)$ in the SE case and $\psi(x) = o(x)$ in the SN case. Since ψ is Beurling φ-equivarying with φ, by Theorem 1, as ψ is measurable

$$\psi(x+u\varphi(x))/\varphi(x) \to 1, \text{ loc unif. in } u.$$

In particular, since $t[\psi(x)/\varphi(x)] \to t$, one has as before

$$\psi(x+t\psi(x))/\psi(x) = \psi(x+t[\psi(x)/\varphi(x)]\varphi(x))/\varphi(x) \cdot \varphi(x)/\psi(x) \to 1.$$

So $\psi \in WSE$ with limit $\lambda = 1$. But ψ is continuous, so by Th. 4 $\psi \in SE$.

As to role reversal here, similarly to Prop. 1, both terms on the right below tend to 1 locally uniformly in t as $x \to \infty$:

$$\varphi(x+t\psi(x))/\psi(x) = \varphi(x+t[\psi(x)/\varphi(x)]\varphi(x))/\varphi(x) \cdot \varphi(x)/\psi(x) \to 1,$$

as $\varphi \in SE$ by the opening remark of the proof. $\square$

Remark. Above, if one assumed instead that $\psi \in WSE$ with limit λ_ψ and as before that ψ is Beurling φ-equivarying with φ, then for any fixed $u > 0$

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x)}{\psi(x+u\varphi(x))}\frac{\psi(x+u\varphi(x))}{\psi(x)} \to \lambda_\psi(u),$$

implying that $\lambda_\psi(u)$ is constant. From here continuing the proof as above yields $\psi \in SE$ with limit λ_ψ, so that $\lambda_\psi = 1$, i.e. ψ is Beurling-slow.

6. Continuous Beurling functions

In this section we prove that every continuous Beurling function is differentiable, provided that λ is continuous and satisfies $\lambda(t) \geq 1$ near the origin. Our approach is via a discrete analogue of the obvious differentiation approach to solving (GS), using constancy of $\Delta_u^\lambda \kappa(x)$. First we clarify the continuity and differentiability conditions of Theorem 0.

Lemma 3. *For $\lambda > 0$ satisfying (GS), if λ is continuous at some point, then it is continuous at all points. Similarly, if λ is differentiable at some point t, then it is differentiable at all points.*

Proof. From (GS) for $u \neq 0$ and fixed t one has the 'Δ^λ-identity'

$$\frac{1}{\lambda(t)}\Delta_u^\lambda \lambda(t) = \frac{\lambda(t+u\lambda(t))-\lambda(t)}{\lambda(t)} = \lambda(u)-1. \qquad (\Delta^\lambda)$$

The linear increasing map $y(u) := t+u\lambda(t)$ carries any open neighbourhood of $u=0$ to an open neighbourhood of t, and likewise for its inverse. Equivalence of global continuity and continuity at $u=0$ follows from this identity (since $\lambda(0)=1$). As to differentiability, the argument is almost the same (upon division by $u \neq 0$). □

The following recurrence occurs in [GolS, Lemma 7], [Bec2] and [Blo].

Definition. For $u>0$ and $\varphi>0$ define the *Beck φ-sequence* $t_n(u)$ in $\mathbb{R}_+$ by the recurrence $t_n := T_u^\varphi(t_{n-1}) = t_{n-1} + u\varphi(t_{n-1})$ with $t_0 = 0$. (Though we do not assume φ monotone, this generalizes the *Beck iteration* of $\gamma(x) := T_1^\varphi(x) = x+\varphi(x)$ via $\gamma_{n+1}(x) = \gamma_1(\gamma_n(x))$, used in bounding flows – see [Bec2, 1.64]; cf. [Blo] or BGT §2.11 and [BinO7, §6]). Call the Beck sequence a *Bloom partition* if $t_n(u)$ diverges, in which case define the *Beck u-step norm* of T (u-step distance from the origin) to be the integer $n = n_T(u)$ such that

$$t_n(u) \leq T < t_{n+1}(u).$$

Lemma 4 (Differencing Lemma). *For λ a Beurling function and $t_n = t_n(u)$ its Beck λ-sequence above,*

$$\kappa(t_n) = \kappa(t_n) - \kappa(t_0) = n\kappa(u), \text{ i.e. } \lambda(t_n) = \lambda(u)^n.$$

Proof. From (GS) one has $\kappa(t_{i+1}) - \kappa(t_i) = \kappa(t_i + u\lambda(t_i)) - \kappa(t_i) = \kappa(u)$. Summing and using $\kappa(0)=0$ yields $\kappa(t_n) = n\kappa(u)$. □

The following, though quite distinct, resembles a result due to Beck [Bec2, 1.69] and relies on (14), a formula noted also in [GolS, Lemma 8].

Proposition 4 (Bounding Formula). *For λ a Beurling function and its associated Beck sequence defined by $t_n := t_{n-1} + u\lambda(t_{n-1})$ with $u>0$, if $\lambda(u) \neq 1$, then*

$$t_n(u) = u\frac{\lambda(u)^n - 1}{\lambda(u)-1} = (\lambda(u)^n - 1) \Big/ \frac{\lambda(u)-1}{u}\ . \qquad (14)$$

Furthermore, for $T, \varepsilon > 0$, if λ is continuous, then for all small enough $u > 0$ and $n = n_T(u)$, the Beck u-step norm of T,

$$-\varepsilon \frac{\lambda(u)^{n+1}}{\lambda(u)^{n+1}-1} < \frac{\lambda(T)-1}{T} - \frac{\lambda(u)-1}{u} < \varepsilon \frac{\lambda(u)^n}{\lambda(u)^n - 1}. \tag{15}$$

Proof. Backwards induction establishes a recurrence for $a_k = a_k(u)$ with

$$t_n = t_{n-k} + a_k(u)\lambda(t_{n-k}), \text{ and } a_0 = 0.$$

Indeed $a_1(u) = u$, and $t_n = t_{n-1} + u\lambda(t_{n-1})$ leads via (GS) to

$$a_{k+1} = a_k \lambda(u) + u.$$

For $\lambda(u) \neq 1$ the recurrence has invariant solution $u/(1-\lambda(u))$, yielding (14). Note that $t_n - t_{n-1} = u\lambda(u)^{n-1} > u$, for $\lambda(u) > 1$.

Now fix $T, \varepsilon > 0$. As κ is assumed continuous at T there is $\delta_\varepsilon > 0$ such that for each t with $|t-T| < \delta_\varepsilon$ one has $|\kappa(T)-\kappa(t)| < \varepsilon$. For any $0 < u < \delta_\varepsilon$, put $n := n_T(u)$. As $|t_n(u) - T| \leq u < \delta_\varepsilon$, one has

$$|\kappa(t_n(u)) - \kappa(T)| < \varepsilon/3.$$

That is, $n\kappa(u) - \varepsilon < \kappa(T) < n\kappa(u) + \varepsilon/3$. So $n \log \lambda(u) - \varepsilon < \log \lambda(T) < n \log \lambda(u) + \varepsilon/3$, or, since $1 - \varepsilon < e^{-\varepsilon}$ and $e^{\varepsilon/3} < 1 + \varepsilon$ (for all small enough ε), one has

$$\frac{(1-\varepsilon)\lambda(u)^n - 1}{T} < \frac{\lambda(T)-1}{T} < \frac{(1+\varepsilon)\lambda(u)^n - 1}{T}.$$

Approximating T from below and above by t_n and t_{n+1} gives

$$\frac{(1-\varepsilon)\lambda(u)^{n+1} - 1}{\lambda(u)^{n+1}-1} \frac{\lambda(u)-1}{u} < \frac{\lambda(T)-1}{T} < \frac{(1+\varepsilon)\lambda(u)^n - 1}{\lambda(u)^n - 1} \frac{\lambda(u)-1}{u},$$

after some rearrangements, yielding (15).

Theorem 6. *If λ is Beurling and continuous, then λ is differentiable (and so of form $\lambda(t) = 1 + at$).*

Proof. We shall prove that $(\lambda(u) - 1)/u$ has a limit as $u \to 0$, i.e. λ is differentiable at the origin and so everywhere, by Lemma 3. For the purposes of this proof only, call a sequence u_n *nice* if it is null (i.e. satisfies $u_n \to 0$), and $\lambda(u_n) \leq 2$ for all n. By continuity of λ at 0, any null sequence may be assumed to be nice.

We claim that for every nice sequence u_n the corresponding quotient sequence $(\lambda(u_n) - 1)/u_n$ is bounded. Otherwise, there is a nice sequence u_n with $\{(\lambda(u_n) - 1)/u_n\}$ unbounded. Take $T = 1$ and let $\varepsilon > 0$ be arbitrary. Without loss of generality suppose that $(\lambda(u_n) - 1)/u_n > 2$.

For $m = m(n) = n_T(u_n)$, as $t_m(u_n) \leq T < t_{m+1}(u_n)$, by Proposition 3

$$\lambda(u_n) \leq \lambda(u_n)^{m(n)} \leq 1 + \frac{\lambda(u_n) - 1}{u_n} \leq \lambda(u_n)^{m(n)+1}.$$

As $\{(\lambda(u_n) - 1)/u_n\}$ is unbounded, so is $\lambda(u_n)^{m(n)+1}$ and $\lambda(u_n)^{m(n)}$ (as $\lambda(u_n) > 1$). By (15), as $\lambda(u_n)^{m(n)} \geq \frac{1}{2}(1+2) = 3/2$ for n large enough,

$$-3\varepsilon < \frac{\lambda(1) - 1}{1} - \frac{\lambda(u_n) - 1}{u_n} < 3\varepsilon,$$

a contradiction to the unboundedness assumption.

Now we may suppose, by passing to a subsequence if necessary, that for every nice sequence u_n the corresponding quotient sequence $(\lambda(u_n) - 1)/u_n$ is not only bounded but in fact convergent. If the limit of the quotient sequence is 0 for each nice sequence, then $\lambda'(0) = 0$, so by the Δ^λ-identity of Lemma 3, $\lambda'(t) = 0$ for all t; then $\lambda(t)$ is constant (and so equal to 1). If, however, the limit of the quotients is not always zero, then fix a nice sequence u_n with positive quotient limit ρ.

Next fix any $T > 2/\rho > 0$. Let $\varepsilon > 0$ be arbitrary. Again take $m = m(n) = n_T(u_n)$. As $\lambda(u_n)^{m(n)} \geq \frac{1}{2}(1+\rho T) \geq 3/2$, for large enough n we have (as when considering unboundedness) that again

$$-3\varepsilon < (\lambda(T) - 1)/T - (\lambda(u_n) - 1)/u_n < 3\varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so

$$(\lambda(T) - 1)/T = \lim_{n\to\infty}(\lambda(u_n) - 1)/u_n = \rho, \text{ i.e. } \lambda(T) = 1 + \rho T.$$

But this holds for all $T > 2/\rho$, making λ differentiable with derivative ρ in $(2/\rho, \infty)$ and so everywhere, including the origin, by Lemma 3. $\square$

Remark. By Proposition 3, $nu \leq t_n(u) \leq T$ for $n = n_T(u)$, so $u \leq T/n$. So if $\lambda(t) = 1 + at$, then $\lambda(u)^n \leq (1 + aT/n)^n \to e^{aT}$ as $u \to 0$, explaining why the unbounded case in the proof above does not arise.

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